ISRAEL JOURNAL OF MATHEMATICS 88 (1994), 189–212

ON THE COMPLEMENT OF THE MANDELBROT SET

BY

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ABSTRACT

We study the uniformization function of the Mandelbrot set via the behavior of multipliers of periodic orbits

Introduction

Let us consider the quadratic family

$$f_c\colon z \mapsto z^2 + c$$

with complex parameter c. Every $c \in \mathbb{C}$ represents a dynamical system with discrete time; that is, it corresponds to a semigroup $\{f_c^n\}_{n=0}^{\infty}$ generated by the map $f_c\colon \mathbb{C} \to \mathbb{C}$. By f^n, we mean the n-fold composition $\underbrace{f \circ \cdots \circ f}_{n}$. The **Julia set** J_c of f_c is the closure of the repelling periodic points of f_c. As is well known, the Julia set is connected if and only if iterates of the critical point of f_c are bounded. The set of such c values

$$M = \{c\colon J_c \text{ is connected}\}$$

is called the **Mandelbrot set**, and is known to be compact. The focus of the present work are maps f_c with disconnected Julia set J_c (i.e. $c \in \mathbb{C} \setminus M$). All such maps are pairwise quasiconformally conjugate.

Fix for a moment $c \in \mathbb{C} \setminus M$. The Julia set J_c is the boundary of the basin of infinity

$$A_c = \{z\colon f_c^n(z) \to \infty \text{ as } n \to \infty\}$$

Received May 18, 1992 and in revised form October 31, 1993

(we will often omit the subscript c if the value of c is clear from the context). In our case $J = \mathbb{C} \setminus A$ and $0 \in A$. There exists a univalent function B which carries a neighborhood of infinity to a neighborhood of infinity and satisfies the functional equation

$$B \circ f(z) = (B(z))^2, \quad B(z) \sim z \text{ at } \infty.$$

The function B is known as the **Böttcher** coordinate function.

The function

$$u(z) = \log |B(z)|$$

extends to a continuous function on the whole complex plane which is harmonic on A and equal to zero on the complement J; in fact, $u(z)$ is the Green's function of A with pole at ∞. For every $r > 0$, we let $G(r) = \{z\colon u(z) > r\}$. The domain

$$G_0 = G(u(0))$$

plays a special role: if $r \geq u(0)$, the domain $G(r) \cup \infty$ is simply-connected in the Riemann sphere, and the map B extends to a univalent function in G_0 whose image is $\{w\colon |w| > u(0)\}$. Note that $0 \in \partial G_0$, and $c \in G_0$ [DH1].

Let us now define the **external rays in the dynamical plane** for $z \mapsto f(z)$ [DH2], [GM], [LS]. For more information, see Section 2 of the present paper. For each $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the set

$$R_t = B^{-1}(\{e^{r+2\pi it}\colon r > u(0)\})$$

is the external ray of angle t in the domain G_0. We extend the external rays from G_0 up to the Julia set as follows: A **smooth** external ray R of f is a maximal C^1-curve in A which orthogonal to the level curves

$$\Gamma(r) = \{z\colon u(z) = r\}$$

for every $r > 0$, and hence the limit set of R belongs to J. An arbitrary external ray is either smooth one, or the limit curve of a sequence of smooth external rays. The **angle** t of a ray $R = R_t$ is the angle of the restriction of R to G_0. When the Julia set is totally disconnected, the ray R_t has a unique limit point $z = z_t \in J$ (called the **landing point** of R_t, or the point of **external argument** t), and conversely, every point $z \in J$ is a landing point of some ray R_t. Given $z \in J$, let

$\Lambda(z)$ be the set of all external arguments of z. The set $\Lambda(z)$ is a compact subset of the circle $\mathbb{T}$. Note that if $t \in \Lambda(z)$, we have $\sigma_2(t) \in \Lambda(f(z))$, where

$$\sigma_2(t) = 2t \,(\mathrm{mod}\, 1).$$

We can also define **external rays** in $\mathbb{C} \setminus M$, as follows [DH1],[DH2]. For each $c \in \mathbb{C} \setminus M$, the complex number

$$B_c(c) = e^{2\pi(h_c + it_c)}$$

is well defined, with $h_c > 0$, $t_c \in \mathbb{T}$. Here, $h_c = u(c)/2\pi$ and t_c is the external angle of the critical value $c = f_c(0) \in G_0$. We will refer to h_c and t_c as the **natural parameters of** c for $c \in \mathbb{C} \setminus M$. A theorem of Douady and Hubbard [DH1] states that correspondence

$$\Phi\colon c \mapsto B_c(c)$$

is the Riemann map of $\mathbb{C} \setminus M$ onto the $\{|w| > 1\}$; in particular, every f_c with disconnected J_c is uniquely determined by the natural parameters $h_c > 0$ and $t_c \in [0, 1)$. The smooth curve

$$R_t^M = \Phi^{-1}(\{re^{2\pi it}\colon r > 1\})$$

is called the **external ray of M at angle t.**

A **hyperbolic component** W of the Mandelbrot set is a connected component of int M for which the map f_c has an attracting cycle $\bar{\alpha}(c)$ of some period m for all $c \in W$ (the attracting cycle of f_c in $\mathbb{C}$ is unique if it exists). The correspondence between c and the multiplier $\lambda(c)$ of this cycle gives the Riemann map $W \mapsto \mathbb{D} = \{|\lambda| < 1\}$ (theorem of Douady–Hubbard–Sullivan [D1]). The function $c \mapsto \lambda(c)$ extends to a homeomorphism $\partial W \to \partial \mathbb{D}$, and defines the **internal argument** (or **internal angle**) of a point $c \in \partial W$: If $\lambda(c) = e^{2\pi i\nu}$, with $\nu \in [0, 1)$, then the internal argument of c is ν. The point c_W with internal argument zero is called the **root** of the hyperbolic component W.

Consider a one-parameter family of maps f_c such that the parameter c leaves $\overline{W}$ through a point $c_* \in \partial W \setminus c_W$. Then the cycle $\bar{\alpha}(c)$ becomes repelling, and the internal argument ν of c_* turns into a combinatorial characterization of the cycle $\bar{\alpha}(c)$. This is the combinatorial rotation number of the cycle $\bar{\alpha}(c)$ (see the definition later on).

Roughly speaking, we describe the connection between the multiplier $\lambda(c)$ and the rotation number ν of the cycle $\bar{\alpha}(c)$ and the natural parameters t_c and h_c of c, when c passes from the hyperbolic component W to the complement $\mathbb{C} \setminus M$ of the Mandelbrot set. Inequalities we will prove are "natural extensions" of **Yoccoz inequality** [Y] . Our argument is based on the concepts of **rotation number** [Y], [GM], and **hedgehog** [SY], [LS].

The Yoccoz inequality (in the case of quadratic family) is related to the case when c leaves $\overline{W}$ through a point $c_* \in \partial W$ with rational internal argument $\nu = \frac{p}{q}$ (in reduced form) and c continues to stay inside the Mandelbrot set. Then [Y]

$$\left|\log \lambda_c - \left(\frac{m \log 2}{q} + \frac{2\pi i p}{q}\right)\right| < \frac{m \log 2}{q},$$

where λ_c is the multiplier of the cycle $\bar{\alpha}(c)$ of period m. Our results are related to the case when c leaves $\overline{W}$ through any point of $\partial W \setminus c_W$. We use the convergents to the continued fraction expansion of the number ν. To every convergent $\frac{p_j}{q_j}$ of ν there corresponds an interval $[h^{(j+1)}, h^{(j)}]$ of the parameter h_c and an interval T_j of the parameter t_c such that

$$\left|\log \lambda_c - \left(\frac{m\aleph}{q_j} + \frac{2\pi i p_j}{q_j}\right)\right| < \frac{m\aleph}{q_j},$$

with some constant $\aleph$ which depends on mq_j and $m(q_{j+1} - q_j)$. Asymptotically, $\aleph \sim 4 \log 2$ and $h^{(j)} \sim T_j$ for $m(q_{j+1} - q_j)$ and mq_j large.

In order to state clearly our results, we start with the Douady–Hubbard description of the external arguments of the hyperbolic components of M [DH1], [D2]. First, consider the main hyperbolic component

$$W_0 = \{c\colon f_c \text{ has an attracting fixed point } \alpha(c)\}.$$

As c goes around ∂W_0, the multiplier $\lambda_0(c)$ of the $\alpha(c)$ goes around $\partial\mathbb{D}$ one time. If

$$\lambda_0(c) = e^{2\pi i \nu}$$

with $\nu = \frac{p}{q} \neq 0$ rational, the corresponding point $c \in \partial W_0$ is a landing point of two external rays of M with arguments

$$\theta^-_{0,1}(\nu) \quad \text{and} \quad \theta^+_{0,1}(\nu).$$

These rays divide $\mathbb{C}$ into two parts. Let c belong to the part not containing W_0. Then one fixed point of f_c has the external argument zero, and the other fixed point $\alpha(c)$ has exactly q external arguments. More exactly, $\Lambda(\alpha(c))$ is a period q cycle of $\sigma_2\colon \mathbb{T} \to \mathbb{T}$, and the points $\theta^{\pm}_{0,1}(\nu)$ are the closest points of this cycle. Moreover, $t_c \in [\theta^{-}_{0,1}(\nu), \theta^{+}_{0,1}(\nu)]$.

If $\lambda_0(c) = e^{2\pi i\nu}$ with ν irrational, then the point $c \in \partial W_0$ is a landing point of exactly one external ray $R^M_{\theta_{0,1}(\nu)}$. If $c \in R^M_{\theta_{0,1}(\nu)}$, then we have $\theta_{0,1}(\nu) = t_c$ and t_c is an external angle of the fixed point $\alpha(c)$.

In both cases, the set $\Lambda(\alpha(c))$ of external arguments of the fixed point $\alpha(c)$ is a **closed minimal rotation set of the map $\sigma_2\colon \mathbb{T} \to \mathbb{T}$, with rotation number ν**. This set is uniquely determined by $\nu \in (0,1)$ [V],[BS].

Now let W be any hyperbolic component such that for each $c \in W$, the map f_c has a stable cycle of period m with multiplier $\lambda(c)$. The root c_W is a landing point of two external rays of M with angles

$$\frac{a}{2^m - 1} \quad \text{and} \quad \frac{b}{2^m - 1},$$

for some "digits" $a, b \in \{1, 2, \ldots, 2^m - 2\}$, with $a < b$ (we assume $W \neq W_0$, i.e. $m \geq 2$).

According to [D2], there is one-to-one correspondence between the external arguments of the points $\partial W_0 \smallsetminus \{c_{W_0}\}$ and those in $\partial W \smallsetminus \{c_W\}$. The algorithm for producing the correspondence is as follows: *take the binary expansion of the external angles $\theta^{\pm}_{0,1}(\nu)$ or $\theta_{0,1}(\nu)$ corresponding to a point c in ∂W_0, replace the digit 0 with a, and replace each 1 with b to obtain a base 2^m expansion of the external arguments $\theta^{\pm}_{a,b}(\nu)$ or $\theta_{a,b}(\nu)$ for the point $\psi(c)$ in ∂W which corresponds to c.*

In fact, the points $\theta^{\pm}_{a,b}(\nu)$ (ν rational) and the point $\theta_{a,b}(\nu)$ (ν irrational) can be characterized as specific points of the unique minimal closed rotation set of the map σ_2^m which has rotation number ν and contains only the digits a and b in the 2^m-base expansion of its points. All such sets are described in [V],[BS] (see Section 3 of the present paper). In particular, the points $\theta^{\pm}_{a,b}(\nu)$ (ν rational) are the closest points of the corresponding finite rotation set.

Remark on the rotation number and on the method of the proof. Let α be a repelling fixed point of a polynomial P. Yoccoz obtains his inequality using the rotation number of α when the Julia set of P is connected. Then the rotation number ν is defined, rational, and describes the order of permutations of classes of homotopic (with fixed ends α and infinity) accesses to α from the basin of infinity D of P [D1], [Y]. One can say that in this case the rotation number does not depend on an invariant foliation Γ transverse to the level curves of the Green's function of D (the classical foliation is orthogonal to the level curves: this foliation is just the set of external rays).

The situation is changed if the Julia set of P is not connected: the rotation number does depend on the foliation Γ. In particular, there exists an access to α from D which is invariant under some iterate P^l [EL]. This means that for an appropriate foliation Γ, the rotation number of α is rational.

We will deal with quadratic family f_c and with foliations which cross the level curves $u(z)$ = const at a fixed angle τ. The foliation Γ^τ is formed as follows: on the domain $f(G_0) \setminus G_0$, let Γ_0 be the family of arcs γ which cross each level curve at angle τ, and then Γ^τ is the collection of all images and preimages of Γ_0 under f. The slope τ determines also the hedgehog, as follows: we make the cuts along the preimages of an arc of a leaf of Γ^τ from the critical value c up to the Julia set and extend the Böttcher function B to a univalent function B^τ of the resulting domain; the boundary of the image is the hedgehog. This is done in [LS] for $\tau = \pi/2$. For an arbitrary τ we obtain in this way a "slanting" τ-hedgehog (see Section 2). The image $B^\tau(R^\tau)$ of a leaf $R^\tau \in \Gamma^\tau$ crosses the unit circle at some point t which is the τ-argument (angle) of R^τ. A periodic point of f_c has ompact set of external τ-arguments and a well-defined rotation number with pect to Γ^τ.

Fhe idea of our method is as follows. Given natural parameters t_c and h_c of we try to choose a foliation Γ^τ in such a way that a given periodic point of has the points $\theta^{\pm}_{a,b}(\nu)$ as its external τ-arguments and so that the interval h the ends $\theta^{\pm}_{a,b}(\nu)$ has an "angle of vision" from the point $t_c + ih_c$ which is too small. Then we apply a general Yoccoz-type inequality (the so-called *ic inequality*, see Section 5) to estimate the multiplier. Particular cases of this inequality corresponding to the angle $\tau = \pi/2$ in our notations have been obtained in [LS] and independently in [G].

We will realize this method of "slanting" hedgehogs for the periodic points satisfying the conditions (C1)–(C4) of the next Section.

ACKNOWLEDGEMENT: I would like to express my gratitude to D. Sullivan and J. Milnor for useful discussions. I am grateful to Pierrette Sentenac for the opportunity to become acquainted with the preprint [BS]. The numerous remarks of referees helped to make the paper much more readable. I thank them very much.

Notation

- $H = \{\omega \in \mathbb{C}\colon \operatorname{Im}\omega > 0\}$, $\bar{H} = H \cup \mathbb{R}$.
- $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, $\mathbb{D}^* = \{z \in \mathbb{C}\colon |z| > 1\}$.
- $\mathbb{T} = \partial\mathbb{D} \simeq \mathbb{R}/\mathbb{Z} \simeq [0,1)$.
- $\sigma_d\colon [0,1) \to [0,1)$, $\sigma_d(t) = d.t(\operatorname{mod} 1)$, $d > 1$.
- $\varrho_d\colon \mathbb{C} \to \mathbb{C}$, $\varrho_d(z) = d.z$, $d > 1$.
- $\tilde{\varrho}_d(x + iy) = \sigma_d(x) + id.y$, $d > 1$, $(x,y) \in [0,1) \times \mathbb{R}$.
- Given $\nu \in (0,1)$, let $\nu = [a_1, a_2, a_3, \dots]$ be the unique continued fraction expansion, with the last element equal to 1 if ν is rational; let

$$\frac{p_j}{q_j} = [a_1, a_2, \dots, a_j]$$

be the convergents to ν, where $j = 1,2,3,\dots$ if ν irrational, and $j = 1,2,3,\dots,N$ with a finite $N = N(\nu)$ if ν rational; in the latter case, $\nu = \frac{p_N}{q_N}$, and we set $q_{N+1} = \infty$.
- For the Green's function $u(z)$ of $f = f_c$ and for $r > 0$, let

$$G(r) = \{z\colon u(z) > r\}, \quad K(r) = \mathbb{C} \setminus G(r), \quad \Gamma(r) = \partial K(r).$$

1. Main results

Let $c_0 \in \mathbb{C} \setminus M$ and $\bar{\alpha}^0 = \{\alpha_i^0\}_{i=1}^m$ be a repelling periodic orbit of f_{c_0} of a period m. Let E be a simply connected domain of $\mathbb{C}$ which is bounded by two external rays of M and a part of the boundary ∂M, such that $c_0 \in E$. Denote by $\bar{\alpha}(c) = \{\alpha_i(c)\}_{i=1}^m$ the m different holomorphic in E functions such that for every $c \in E$, $\bar{\alpha}(c)$ is a periodic orbit of f_c with period m, and $\alpha_i(c_0) = \alpha_i^0$, $i = 1, \dots, m$. The continuation $\bar{\alpha}(c)$ of the cycle $\bar{\alpha}^0$ is defined uniquely in the following sense: if $\bar{\alpha}_1(c)$ and $\bar{\alpha}_2(c)$ are two continuations corresponding to domains E_1 and E_2 as above, then $\bar{\alpha}_1(c) \equiv \bar{\alpha}_2(c)$ in the common part of E_1 and E_2 which contains c_0.

From now on we fix c_0, $\bar{\alpha}^0$ and hence the $\bar{\alpha}(c)$ as above. Note that

$$\alpha_i(c) \neq \alpha_j(c), \quad i \neq j,$$

since f_c has no neutral cycles if $c \in \mathbb{C} \setminus M$.

Given c, let us consider the set of external arguments $\Lambda(\alpha_i)$ of the point $\alpha_i = \alpha_i(c)$, $i = 1, \dots, m$, and let

$$\Lambda(\bar{\alpha}) = \bigcup_{i=1}^{m} \Lambda(\alpha_i).$$

Every $\Lambda(\alpha_i)$ is a non-empty proper compact subset of $\mathbb{T}$, and the map σ_2 permutes the sets $\Lambda(\alpha_i)$, $i = 1, \dots, m$. On the other hand, the map f^m preserves the cyclic order of the external rays landing at α_i, allowing us to define the rotation number as follows [Y], [GM].

Let $d = 2^m$. The set $\Lambda(\bar{\alpha})$ is the **rotation set of the map σ_d with rotation number ν.**

Definition 1.1: A compact $\Lambda \subset \mathbb{T}$ is called the **rotation set** of the map σ_d if $\sigma_d(\Lambda) \subseteq \Lambda$ and the restriction $\sigma_d|_\Lambda$ can be extended to a map of $\mathbb{T}$ to $\mathbb{T}$ which lifts to a non-decreasing continuous map $F\colon \mathbb{R} \to \mathbb{R}$ such that $F - id$ is 1-periodic.

Under these conditions, the limit

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}, \quad x \in \mathbb{R},$$

exists, and its fractional part ν depends only on (σ_d, Λ). The number $\nu \in [0, 1)$ is called the **rotation number of the set Λ.**

The sets $\Lambda(\alpha_i)$, $i = 1, \dots, m$, are also rotation sets of σ_d with the same rotation number ν. The number ν is either irrational ($\Lambda(\bar{\alpha})$ is infinite) or rational ($\Lambda(\bar{\alpha})$ is finite) [GM]. In the latter case, $\Lambda(\bar{\alpha})$ consists of σ_d-cycles of period q, where $\nu = p/q$ (in reduced form). We will assume $\nu \neq 0$.

We will say the cycle $\bar{\alpha} = \bar{\alpha}(c)$ satisfies the **two-digit conditions**, if the following conditions (C1)–(C4) are fulfilled:

(C1) there exists a point $\theta \in \Lambda(\bar{\alpha})$, which has a d-expansion containing only two digits $a, b \in \{0, 1, \dots, d-1\}$, with $a < b$;

(C2) if the period m is not 1, then $a \neq 0$ and $b \neq d - 1$;

(C3) if $\Lambda(\bar{\alpha})$ is infinite (ν irrational), then $\theta = t_c = \theta_{a,b}(\nu)$;

(C4) if $\Lambda(\bar{\alpha})$ is finite (ν rational), then θ is one of the two closest points $\theta^-_{a,b}(\nu)$, $\theta^+_{a,b}(\nu)$ of the finite σ_d-orbit of θ, and $t_c \in [\theta^-_{a,b}(\nu), \theta^+_{a,b}(\nu)]$.

Remark 1.1: If the cycle $\bar{\alpha}^0$ is a fixed point and $\nu^0 \neq 0$, the conditions (C1)–(C4) hold with $a = 0, b = 1$ [DH1]. One can consider this fact, together with the Douady–Hubbard description of the external arguments of the hyperbolic components M, as justification to introduce these conditions.

For fixed c_0 and corresponding cycle $\bar{\alpha}^0 = \bar{\alpha}(c_0)$, we consider the rotation number $\nu^0 \neq 0$ of this cycle and the corresponding convergents to ν^0: $\{\frac{p_j}{q_j}\}_{j=1}^N$ (see Notation). In particular, $\nu^0 = \frac{p_N}{q_N}$, if $N < \infty$ (i.e. ν^0 rational).

Fix any finite j, such that

$$j = 1, 2, \dots, \text{ if } N = \infty, \quad \text{and} \quad j = 1, 2, \dots, N, \text{ if } N < \infty.$$

Define a domain Ω_j in the complex plane as follows. First, let

$$h^{(j)} = \frac{(b-a)(2^m - 1)}{2^{mq_j}}$$

(for convenience, we set $q_{N+1} = \infty$ (i.e. $h^{(N+1)} = 0$) if $j = N < \infty$). Second, define an interval

$$T_j = \begin{cases} [\theta^-_{a,b}(\frac{p_{j+1}}{q_{j+1}}), \theta^+_{a,b}(\frac{p_j}{q_j})], & \text{if } \frac{p_{j+1}}{q_{j+1}} < \frac{p_j}{q_j},\ j \neq N, \\ [\theta^-_{a,b}(\frac{p_j}{q_j}), \theta^+_{a,b}(\frac{p_{j+1}}{q_{j+1}})], & \text{if } \frac{p_{j+1}}{q_{j+1}} > \frac{p_j}{q_j},\ j \neq N, \\ [\theta^-_{a,b}(\frac{p_N}{q_N}), \theta^+_{a,b}(\frac{p_N}{q_N})], & \text{if } j = N < \infty. \end{cases}$$

Now, let Ω_j be the rectangle

$$\Omega_j = \{t + ih \colon h^{(j+1)} \leq h \leq h^{(j)}, t \in T_j\}.$$

Remark 1.2: The main property we need from the set Ω_j is that the angle of vision of the interval $[\theta^-_{a,b}(\frac{p_j}{q_j}), \theta^+_{a,b}(\frac{p_j}{q_j})] \subset \mathbb{R}$ is bounded from below by an absolute positive constant when viewed from any point $\omega \in \Omega_j$.

Remark 1.3: The length of the interval T_j is given by

$$|T_j| = \frac{(b-a)(2^m - 1)}{(2^{mq_j} - 1)} \cdot \frac{(2^{mq_{j+1}} + 2^{mq_j} - 1)}{(2^{mq_{j+1}} - 1)}$$

(if $j = N$, the second factor disappears). In particular,

$$\frac{|T_j|}{h^{(j)}} \to 1, \quad \text{as } mq_j \to \infty \text{ and } m(q_{j+1} - q_j) \to \infty.$$

We assume also that if the period $m \geq 2$, then $q_j \geq 4$, i.e.

$$\frac{p_j}{q_j} \neq \frac{1}{2}, \frac{1}{3}, \frac{2}{3}.$$

THEOREM 1.1: *If the cycle $\bar{\alpha}^0 = \bar{\alpha}(c_0)$ satisfies the two-digit conditions* (C1)–(C4), *then*

(A) *the cycle $\bar{\alpha}(c)$ satisfies the two-digit conditions* (C1)–(C4) *for every c with natural parameters $h_c > 0$ and $t_c \in T_j$;*

(B) *there exists a bounded function $\aleph$ of two variables,*

$$\aleph = \aleph(mq_j, m(q_{j+1} - q_j)),$$

such that for some branch $\log \lambda_c$ of the logarithm of the multiplier λ_c of $\bar{\alpha}(c)$, we have

$$\log \lambda_c \in D\Big(\frac{p_j}{q_j}, m, \aleph\Big) := \left\{z\colon \left|z - \left(\frac{m\aleph}{q_j} + \frac{2\pi i p_j}{q_j}\right)\right| < \frac{m\aleph}{q_j}\right\}$$

whenever $t_c + ih_c \in \Omega_j$. Moreover,

$$\lim \aleph(mq_j, m(q_{j+1} - q_j)) = 4\log 2$$

as $mq_j \to \infty$ and $m(q_{j+1} - q_j) \to \infty$.

Remark 1.4: Actually we will prove part (A) of the theorem in more general form: see Theorem 4.1.

An immediate consequence of the theorem is the following

COROLLARY 1.1: *Under the conditions of Theorem 1.1, if $t_c \in T_j$ and $h_c \le h^{(j)}$, then $\log \lambda_c$ belongs to the convex hull of the balls $D(\frac{p_j}{q_j}, m, \aleph)$ and $D(\frac{p_{j+1}}{q_{j+1}}, m, \aleph)$.*

In particular, if $N = \infty$ and $j \to \infty$, then $\log \lambda_c \to 2\pi i \nu_0$. This fact, together with the Douady–Hubbard description, yields the following

COROLLARY 1.2: *Let c_0 belong to an external ray $R^M_{t_0}$ of the Mandelbrot set at angle t_0. This ray lands at a unique point c_* of ∂W with an irrational internal angle ν^0 if and only if there exists a cycle $\bar{\alpha}(c_0)$ of f_{c_0}, which satisfies conditions* (C1)–(C3). *Moreover, if $t_c \in T_j$ and $h_c \le h^{(j)}$, $j = 1, 2, \ldots$, then*

$$|\log \lambda_c - 2\pi i \nu^0| < \frac{K_j m}{q_j},$$

where $K_j \to 8 \log 2$ as $j \to \infty$.

Remark 1.5: Note that the distance between ν_0 and one of the ends of the interval T_j is $\sim (b-a)2^{-m(q_j-1)} \sim h^{(j)}$, and the distance between ν_0 and other end is $\sim (b-a)2^{-m(q_{j+1}-1)}$.

Now consider the case of rational rotation number. Let, under the conditions of the theorem,

$$\nu^0 = \frac{p}{q} = [a_1, a_2, \dots, a_N], \quad a_N = 1,$$

with $q \geq 4$ if $m \geq 2$. For every $k \in \mathbb{N}$, let τ_k^- be the smaller of the two numbers

$$[a_1, a_2, \dots, a_N, k] \ , \ [a_1, a_2, \dots, a_{N-1} + a_N, k],$$

and let τ_k^+ be the larger. Then $\tau_k^+ \searrow \frac{p}{q}$, $\tau_k^- \nearrow \frac{p}{q}$ as $k \to \infty$.

Let, for example, N be even. It follows from Corollary 1.1 that if

$$0 < t_c - \theta_{a,b}^+(\frac{p}{q}) \leq \frac{(b-a)(2^m-1)2^{mq}}{2^{mq}-1} \frac{1}{2^{mqk+mq_{N-1}}},$$

$$0 < h_c \leq \frac{2^{mq}-1}{2^{mq}}(t_c - \theta_{a,b}^+(\frac{p}{q})),$$

then

$$\left| \log \lambda_c - \frac{2\pi i p}{q} \right| < \frac{\aleph_0 m}{kq + q_{N-1}} \tag{1.1}$$

where $\aleph_0$ is an absolute constant. To obtain a similar inequality for the left point $\theta_{a,b}^-(\frac{p}{q})$, one needs only to replace q_{N-1} in the above with q_{N-2}. This implies the following

COROLLARY 1.3: *Let the cycle $\bar{\alpha}(c_0)$ satisfy* (C1)–(C4) *with rational rotation number $\nu^0 = \frac{p}{q}$ (in reduced form), and $q \geq 4$ if $m \geq 2$. Then two rays $R^M_{\theta^{\pm}_{a,b}(\nu^0)}$ of M land at a common point c_* of ∂W; this point has internal argument ν^0.*

2. Preliminary notions and constructions

2.1. THE HEDGEHOG (cf. [SY], [LS]). Fix a polynomial $f = f_c$ with disconnected Julia set. Given an angle (slope) $\tau \in (0, \pi)$, we construct the τ-**hedgehog** as follows.

We know that the Böttcher function $B(z)$ is well defined on G_0 and satisfies there the functional equation $B(f(z)) = [B(z)]^2$. This equation yields an analytic (infinitely valued) continuation of the function $B(z)$ on the whole domain A. The continued function has branch points at the points of the set

$$C(\infty) = \bigcup_{m=0}^{\infty} f^{-m}(0).$$

We obtain a **single-valued** analytic extention of $B(z)$ by cutting the domain A along certain lines (which we call τ-**cuts**).

Let $z \in A \setminus C(\infty)$. There exists a unique maximal C^1-curve passing through z which meets any level line $\Gamma(r)$ at the same angle τ. We call this curve the τ-**curve** $R(z)$. The direction of the τ-curve is chosen so that the Green's function $u(z)$ is decreasing along it. The origin of every τ-curve is either ∞ or some point in $C(\infty)$. In the former case the τ-curve is called the τ-**radius**, or **smooth** τ-**ray** (of f), in the latter case it is called the τ-**cut**.

The Böttcher function B extends along every smooth ray. Let A_τ denote the set of points which lie on smooth τ-rays. Then $A_\tau \cup \{\infty\}$ is simply-connected in the Riemann sphere. Its complement J_τ is the Julia set completed by the τ-cuts. The extended univalent function B^τ maps the domain A_τ one-to-one onto a **hedgehog-like** domain U_τ. The boundary $S_\tau = \partial U_\tau$ is called the τ-**hedgehog** or **slanting hedgehog**.

It is convenient to straighten the hedgehog with the help of logarithmic coordinates. Consider exterior $\mathbb{D}^*$ of the unit disk and its universal covering $H = \{\zeta\colon \operatorname{Im}\zeta > 0\}$ with a covering anti-conformal projection $p\colon H \to \mathbb{D}^*$,

$$p\colon \omega \mapsto \exp(2\pi i\bar{\omega}).$$

For every $\omega \in H$ let L_ω be a straight line through ω which intersects the real line at the angle τ. Let $X_\tau(\omega)$ be the point of the intersection. We define

$$\arg_\tau(\omega) = X_\tau(\omega) (\operatorname{mod} 1).$$

The preimage $H_\tau = p^{-1}(U_\tau)$ of the hedgehog-like domain U_τ is a universal covering of U_τ. Moreover,

$$H_\tau = \overline{H} \setminus Q_\tau,$$

where $Q_\tau = p^{-1}(S_\tau)$ is a one-periodic comb, τ-**comb**. The map

$$\Phi = (B^\tau)^{-1} \circ p\colon H_\tau \to A_\tau$$

is an analytic unbranched covering.

Recall that $\sigma_2(x) = 2x (\mathrm{mod}\, 1)$, $\varrho_2\colon \omega \mapsto 2\omega$ and $\tilde{\varrho}_2(x+iy) = \sigma_2(x) + i2y, (x,y) \in [0,1) \times \mathbb{R}$.

PROPOSITION 2.1 (cf. [LS]):

1. $f\colon A_\tau \to A_\tau$, $\varrho_2\colon H_\tau \to H_\tau$, *and*

$$\Phi \circ \varrho_2 = f \circ \Phi \quad in\ H_\tau.$$

2. *The comb*

$$Q_\tau = \partial H_\tau = \Big\{[0,1) \bigcup \bigcup_{n=1}^{\infty} \bigcup_{\tilde{\varrho}_2^n(x)=t_c+ih_c} N_x\Big\} + \mathbb{Z}, \tag{2.1}$$

where $N_x = \{\omega\colon 0 \le \mathrm{Im}(\omega) \le \mathrm{Im}(x), \arg_\tau \omega = \arg_\tau x\}$.

The segments N_x, with x as in (2.1), will be called the τ-**needles** of the comb Q_τ. The ground of this comb is the real axis, and the ends x of the needles of Q_τ are the points with coordinates

$$\omega(n,k) = \frac{t_c + k}{2^n} + i\frac{h_c}{2^n},$$

where $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\exp(2\pi(h_c + it_c)) = B(c)$. We will call the positive integer n here the **level of the needle**. We will call the segment $N_{t_c+ih_c}$ the **generating segment** of the comb Q_τ. Note that all needles are the pre-images of the generating segment under $\tilde{\varrho}_2$ shifted by $\mathbb{Z}$.

Remark 2.1: We consider the anti-conformal projection p instead of the conformal one for convenience, so that the comb Q_τ lies in the upper half-plane. A property we shall need is that the anti-conformal as well as conformal maps do not change modulus of families of curves [Ah].

2.2. THE EXTERNAL RAYS AND ANGLES (cf. [DH2],[GM]). We define the τ-**rays** as follows. Note that usual (orthogonal) external rays correspond to $\tau = \pi/2$. The τ-ray is just a τ-radius R, if R extends up to the Julia set (i.e. R does not end at a point of $C(\infty)$). Let the endpoint of R be a point of $C(\infty)$. Then the full preimage $\Phi^{-1}(R)$ is $L + \mathbb{Z}$, where the straight ray L lands at the top x of some needle N_x. The function Φ extends to two continuous functions on either side of N_x. This allows us to define the two τ-rays corresponding to the τ-radius R as the images of two sides of the curve

$$L \cup N_x = \{\omega \in H\colon 0 < \mathrm{Im}(\omega) < \infty, \arg_\tau \omega = t\}, \quad t = \arg_\tau x;$$

This gives the **right** and **left limit rays** R^+ and R^-.

Every τ-ray R has an **external τ-argument**: this is the $\arg_\tau$ of the points of $\Phi^{-1}(R)$. In particular, the right and left limit rays $R^\pm$ corresponding to the τ-radius R have the same external τ-argument; this is the τ-argument of R.

2.3. Two properties of the hedgehog. The first property is, in fact, a property of the comb Q_τ itself, so we shall forget about the polynomial f and only observe the geometry of Q_τ. The second property is related to the construction of the domain A_τ.

Let $X = \{x_1, \dots, x_n\}$ be a cycle of the map σ_2: $[0,1) \to [0,1)$, and let $\tilde{X} = X + \mathbb{Z}$. Define two values (of angles) $\gamma^{(r)}(X) \in (0, \pi - \tau)$, $\gamma^{(\ell)}(X) \in (0, \tau)$ as follows. Look at the generating segment $N_{t_c+ih_c}$ of Q_τ, and find the point $x^{(r)}$ of the set $\tilde{X}$ which is closest to $N_{t_c+ih_c}$ from the right. Let us consider the triangle $\Delta^{(r)}$ with a vertex $x^{(r)}$ and the opposite side $N_{t_c+ih_c}$. Then the angle $\gamma^{(r)}(X)$ is said to be an angle of $\Delta^{(r)}$ at the vertex $t_c + ih_c$. The angle $\gamma^{(\ell)}(X)$ is defined in an analogous fashion, but using the point $x^{(\ell)}$ of $\tilde{X}$ which is closest to $N_{t_c+ih_c}$ from the left, and the corresponding triangle $\Delta^{(\ell)}$.

Lemma 2.1 (the first property, cf. [LS]): *For every $x \in \tilde{X}$, the angles*

$$W_\tau^{(r)}(x) = x + \{\omega \in H\colon \pi - \tau - \gamma^{(r)}(X) < \arg\omega < \pi - \tau\},$$
$$W_\tau^{(\ell)}(x) = x + \{\omega \in H\colon \tau - \gamma^{(\ell)}(X) < \arg\omega < \tau\}$$

belong to H_τ, and they are the maximal open angles at the top $x \in \tilde{X}$ with this property (here $\arg\omega$ is the standard $\pi/2$-argument of $\omega \in \mathbb{C} \setminus \{0\}$).

If x is not a base of any needle, then the angle

$$W_\tau(x) = W_\tau^{(r)}(x) \bigcup W_\tau^{(\ell)}(x) \bigcup \{\omega \in H\colon \arg_\tau \omega = \arg_\tau x\}$$

of the value $\gamma(X) = \gamma^{(\ell)}(X) + \gamma^{(r)}(X)$ also belongs to H_τ.

Proof: Use that the map $x + iy \mapsto 2.x + i2.y$ acts in H_τ as well as in Q_τ whenever $y \le h_c/2$.

Remark 2.2: The value $\gamma(X)$ is the **angle of vision** of the interval $(x^{(\ell)}, x^{(r)}) \subset \mathbb{R}$ from the point $t_c + ih_c \notin \mathbb{R}$.

To formulate the second property, let us consider the boundary Γ_0 of one component of $\mathrm{int}K(h/2^m)$. Let $\Lambda_0 \subset \mathbb{T}$ be the compact set of external arguments of the τ-rays which cross Γ_0. Fix $t_1, t_2 \in \Lambda_0$ and $t_1 \neq t_2$. These split $\mathbb{T}$ into two open parts; let I_0 be the shortest one.

LEMMA 2.2 (the second property): *Let some point $t_0 \in I_0$ and $t_0 \notin \Lambda_0$. Then the length of I_0 is not less than 2^{-m}.*

Proof: Consider a simply-connected domain $\Omega = A_\tau \cap G(h/2^m)$. Its boundary $\partial\Omega$ consists of the curves $\Gamma(h/2^m)$ joined by τ-cuts. In particular, $\Gamma_0 \subset \partial\Omega$. Let z_0, z_1, and z_2 be the points of intersections of $R^\tau_{t_0}$, $R^\tau_{t_1}$ and $R^\tau_{t_2}$ with $\partial\Omega$. Then we have $z_1, z_2 \in \Gamma_0$, $z_0 \notin \Gamma_0$. Let us go around $\partial\Omega$ from z_1 to z_2 through z_0 (the domain Ω stays on one side). At some point $z_* \in \Gamma_0$, we leave Γ_0 along one side of a cut $\ell_* \ni z_0$ and then return to Γ_0 from another side of ℓ_*. The point $z_* \in \ell_* \cap \Gamma_0$ corresponds to two points t'_* and t''_* in the arc $I_0 = (t_1, t_2)$ such that $|t'_* - t''_*| = \ell/2^k$ for some $\ell \in \mathbb{N}$ and $k \in \{1, 2, \ldots, m\}$. ∎

Remark 2.3: The following will be important. Let $t \in I_0$ be **any** point such that, for some $k = 1, 2, \ldots, m$, t is the base of a needle of the level k. Then t cannot belong to Λ_0 and, hence, the length of I_0 is not less than 2^{-m}. In fact, t is the τ-argument of the left and the right rays, and one of them does not intersect Γ_0.

2.4. THE τ-ROTATION SET AND τ-ROTATION NUMBER OF A PERIODIC POINT. Let a be a point of the Julia set J. The set of all τ-arguments of the τ-rays landing at a is denoted by $\Lambda_\tau(a)$. This is a non-empty, closed, proper subset of $\mathbb{T}$.

Additionally, let a be a repelling fixed point of $P = f^m$. Then $\Lambda_\tau(a)$ is invariant under the action of $\sigma_d\colon \mathbb{T} \to \mathbb{T}$, where $d = 2^m$. Moreover, $\Lambda_\tau(a)$ is a rotation set of σ_d with a rotation number $\nu_\tau(a)$.

3. Rotation sets with two symbols

We call a rotation set Λ of σ_d a **rotation set with two symbols** $a, b \in \{0, 1, \ldots, d-1\}$ if one can write every point $\theta \in \Lambda$ as

$$\theta = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{d^i},$$

with $\varepsilon_i \in \{a, b\}$. We will need the following information on the rotation sets of σ with two symbols. In what follows we fix d and the symbols a and b. Replacement of a, b, and d by 0, 1, and 2 respectively gives a one-to-one correspondence φ between the rotation set of σ_d with symbols a, b and those of σ_2. The rotation sets of σ_2 are well-studied (see [V], [BS] and [DH1]). The following statement easily follows from [BS], with some remarks about the distances (note that the correspondence φ does not preserve distances).

PROPOSITION 3.1 (cf. [V], [BS]): *For each $\nu \in [0, 1)$ there exists a unique closed minimal σ_d-rotation set Λ^ν with two symbols a, b.*

I. *For rational $\nu = \frac{p}{q} \neq 0$ (in reduced form), the set Λ^ν is a σ_d-cycle of period q. If $\nu = 0$, then Λ^0 is $a/(d-1)$ or $b/(d-1)$.*

Let $\nu \neq 0$. There exist uniquely defined rationals

$$\theta_1^-(\nu) \leq \theta^-(\nu) < \theta^+(\nu) \leq \theta_1^+(\nu),$$

such that

(i) *$\theta^-(\nu)$ and $\theta^+(\nu)$ are adjacent and are the closest points of $\Lambda^{p/q}$, the points $\theta_1^\pm(\nu)$ are the extreme points of Λ^ν:*

$$\Lambda^\nu \subset [\theta_1^-(\nu), \theta_1^+(\nu)],$$

and $\sigma_d(\theta_1^+(\nu)) = \theta^-(\nu)$, $\sigma_d(\theta_1^-(\nu)) = \theta^+(\nu)$;

(ii)

$$\Delta(\nu) = \theta^+(\nu) - \theta^-(\nu) = \frac{(d-1)(b-a)}{d^q - 1}, \tag{3.1}$$

and for every two adjoint rationals $\nu = p/q$ and $\nu' = P/Q$, $Qp - Pq = 1$, we have:

$$\theta^-(\nu) - \theta^+(\nu') = \frac{(b-a)(d-1)}{(d^q-1)(d^Q-1)}.$$

II. *For every irrational $\nu \in (0, 1)$ there exists a unique real number $0 < \theta(\nu) < 1$ such that*

(i) *the closure of its σ_d-orbit is Λ^ν;*

(ii) *the extreme points $\theta_1^-(\nu)$ and $\theta_1^+(\nu)$ of Λ^ν are preimages of σ_d:*

$$\Lambda^\nu \subset [\theta_1^-(\nu), \theta_1^+(\nu)], \qquad \sigma_d(\theta_1^\pm(\nu)) = \theta(\nu).$$

III. *If $\nu_0 < \nu_1 < \nu_2$, with ν_0, ν_2 rationals and ν_1 irrational, then*

$$\theta^+(\nu_0) < \theta(\nu_1) < \theta^-(\nu_2).$$

For every $\theta \in (\frac{a}{d-1}, \frac{b}{d-1})$, there exists the unique $\nu \in \mathbb{T}$, such that either $\theta = \theta(\nu)$, with ν irrational, or $\theta \in [\theta^-(\nu), \theta^+(\nu)]$, with ν rational. Moreover, $\nu \in (0,1)$ is a nondecreasing function of θ. (In fact, it is a devil's staircase.)

Remark 3.1: Comparing with the notation at the end of the introduction, we have $\theta^{\pm}_{a,b}(\nu) = \theta^{\pm}(\nu)$ (ν rational), and $\theta_{a,b}(\nu) = \theta(\nu)$ (ν irrational).

We will denote

$$\theta^-(\nu) = \theta^+(\nu) = \theta(\nu) \quad \text{if } \nu \text{ irrational.}$$

Proof of Proposition 3.1: By [BS], lemma 1 (iii), d-expansions of $\theta^{\pm}(\frac{p}{q})$ are

$$\theta^+\left(\frac{p}{q}\right) = 0\,(\varepsilon_1\varepsilon_2\ldots\varepsilon_{q-2}ba) \quad \text{and} \quad \theta^-\left(\frac{p}{q}\right) = 0.\,(\varepsilon_1\varepsilon_2\ldots\varepsilon_{q-2}ab),$$

where $\varepsilon_i \in \{a,b\}$, and the brackets denote a repeated block of symbols. This implies (3.1) and also that $\theta^{\pm}(\frac{p}{q})$ are the nearest points of $\Lambda^{p/q}$. The rest of I(ii) also follows from d-expansions of $\theta^{\pm}\left(\frac{p_{N-1}}{q_{N-1}}\right)$, see [BS].

Remark 3.2: The numbers $\theta^{\pm}(\nu)$,$\theta(\nu)$ are constructed by an explicit algorithm. For instance,

$$\theta^-\left(\frac{1}{2}\right) = 0.(ab), \quad \theta^+\left(\frac{1}{2}\right) = 0.(ba),$$

$$\theta^-\left(\frac{1}{3}\right) = 0.(aab), \quad \theta^+\left(\frac{1}{3}\right) = 0.(aba), \quad \theta^-\left(\frac{2}{3}\right) = 0.(bab), \quad \theta^+\left(\frac{2}{3}\right) = 0.(bba).$$

Let

$$J^{(a)} = \left(\frac{a}{d-1}, \frac{a+1}{d}\right) \quad \text{and} \quad J^{(b)} = \left(\frac{b}{d}, \frac{b}{d-1}\right).$$

A rotation set Λ^ν, $\nu \neq 0$, splits into two non-empty parts Λ^ν_a and Λ^ν_b:

$$\Lambda^\nu_\varepsilon = \Lambda^\nu \cap J^{(\varepsilon)}, \quad \varepsilon \in \{a,b\}.$$

Denote by $I^\nu_a = [\theta^-_1(\nu), \theta^a(\nu)]$ and $I^\nu_b = [\theta^b(\nu), \theta^+_1(\nu)]$ two minimal closed intervals containing Λ^ν_a and Λ^ν_b. It is easy to understand that $\sigma_d(\theta^a(\nu) = \theta^+_1(\nu)$ and $\sigma_d(\theta^b(\nu) = \theta^-_1(\nu)$.

In Section 4 we will use the following. For every $t \in (\frac{a}{d-1}, \frac{b}{d-1})$, there exist unique points $t^{(\ell)} \in J^{(a)}$ and $t^{(r)} \in J^{(b)}$, such that

$$\sigma_d(t^{(\ell)}) = \sigma_d(t^{(r)}) = t.$$

Thus, we can find unique points $t^a \in J^{(a)}$ and $t^b \in J^{(b)}$ for which

$$\sigma_d(t^a) = t^{(r)} \quad \text{and} \quad \sigma_d(t^b) = t^{(\ell)}.$$

The points $t^{(\ell)}$, $t^{(r)}$, t^a, and t^b are increasing functions of t, and $t^{(\ell)} < t^a < t^b < t^{(r)}$. Set

$$L_{a,b}(t) = (t^{(\ell)}, t^a) \cup (t^b, t^{(r)}).$$

Then we have:

$$\theta_1^-(\nu) = (\theta^+(\nu))^{(\ell)}, \quad \theta_1^+(\nu) = (\theta^-(\nu))^{(r)},$$
$$\theta^a(\nu) = (\theta^-(\nu))^a, \quad \theta^b(\nu) = (\theta^+(\nu))^b.$$

Now, if $t \in [\theta^-(\nu), \theta^+(\nu)]$, then $\Lambda^\nu \subset \overline{L_{a,b}(t)}$. Moreover, $I_a^\nu \subset [t^{(\ell)}, t^a]$, $I_b^\nu \subset [t^b, t^{(r)}]$, and we have the equalities iff $t = \theta^-(\nu) = \theta^+(\nu)$, i.e. if ν is irrational.

4. Proof of the theorem: Part (A)

In this section we prove a statement which is more general than Part (A) of Theorem 1.1. Recall that we fixed $d = 2^m$ and the symbols $a, b \in \{0, 1, \dots, d-1\}$.

THEOREM 4.1: *Let the cycle $\bar{\alpha}(c_0)$ satisfy the two-digit conditions* (C1)–(C4). *Let T be an open interval such that $t_{c_0} \in T$ and*

(4.1)
$$T \subset (J^a \cup J^b) \setminus \left(\left[\theta^-\left(\frac{1}{3}\right), \theta^+\left(\frac{1}{3}\right)\right] \cup \left[\theta^-\left(\frac{2}{3}\right), \theta^+\left(\frac{2}{3}\right)\right] \cup \left[\theta^-\left(\frac{1}{2}\right), \theta^+\left(\frac{1}{2}\right)\right] \right).$$

For the natural parameters t_c and h_c of some $c \in \mathbb{C} \setminus M$, and for some slope τ, we have $t_c^\tau = \arg_\tau(t_c + ih_c)$ lying in T, that is, there exists a unique $\nu \in (0,1)$ such that $t_c^\tau = \theta(\nu)$ or $t_c^\tau \in [\theta^-(\nu), \theta^+(\nu)]$. Then the τ-rotation number of the cycle $\alpha(c)$ is equal to ν and the τ-rotation set of $\alpha(c)$ contains Λ^ν.

We shall start to prove this theorem. By condition (C1), the unique minimal closed σ_d-rotation set Λ^{ν^0} with two digits a, b is contained in $\Lambda_{\pi/2}(\alpha_1^0)$, for a point $\alpha_1^0 \in \bar{\alpha}(c_0)$. Here $\nu^0 \in (0,1)$ is the $\pi/2$-rotation number of $\bar{\alpha}(c_0)$. By (C3)–(C4), $t_0 = t_{c_0} \in [\theta_{a,b}^-(\nu^0), \theta_{a,b}^+(\nu^0)]$.

Without loss of generality we may assume $t_0 \in J^{(a)}$. We will say that a point $x \in \mathbb{T}$ has t-**level** k, if $\sigma_2^k(x) = t$.

LEMMA 4.1: *There are no points of t_0-levels $k \le m$ in $L_{a,b}(t_0)$.*

Proof: Suppose some point x of t_0-level $k \le m$ lies in $L_{a,b}(t_0)$. If $x \in (\theta_1^-(\nu^0), \theta^a(\nu^0))$, then $|\theta_1^-(\nu^0) - \theta^a(\nu^0)| \ge 1/2^m$ by Remark 2.3 (following Lemma 2.2). This gives a contradiction. Similarly, $x \notin (\theta^b(\nu^0), \theta_1^+(\nu^0))$. In particular, this proves the lemma if ν^0 is irrational. Now let ν^0 be rational. Assume $x \in (t_0^{(\ell)}, \theta_1^-(\nu^0)]$. Then $y = \sigma_2^k(\theta_1^-(\nu^0)) \in (t_0, \theta^+(\nu^0))$. It follows that $k < m$. Hence the ray $R_y^{\pi/2}$ of the argument y lands at a point of the cycle $\bar{\alpha}^0$ different from α_1^0. It contradicts Lemma 2.2. Assume now that $x \in [\theta^a(\nu^0), t_0^a)$. In this case $y = \sigma_2^k(\theta^a(\nu^0)) \in (\theta^-(\nu^0), t_0)$, and again the ray $R_y^{\pi/2}$ lands at a point of the cycle $\bar{\alpha}^0$ different from α_1^0. Contradiction. The case $x \in (t_0^b, \theta^b(\nu^0)] \cup [\theta_1^+(\nu^0), t_0^{(r)})$ can be handled similarly.

LEMMA 4.2: *For every $t \in T$, there are no points of the t-levels $k \le m$ in $L_{a,b}(t)$.*

Proof: The set of the points of a t-level k are values of 2^k continuous functions on $t \in J^{(a)}$:

$$t_{r,k} = \frac{t+r}{2^k}, \quad r = 0, 1, \dots, 2^k - 1.$$

Assume that for some $t_1 \in T \subset J^{(a)}$ a point of t_1-level $\le m$ is in $L_{a,b}(t_1)$. Let, for example, $t_1 > t_0$. Consider the interval $(t_0, t_1) \subset J^{(a)}$. The set $L_{a,b}(t)$ is open. Hence there exists a $t_* \in (t_0, t_1)$ such that: (1) for $t \in (t_0, t_*]$, there are no points of t-levels $\le m$ in $L_{a,b}(t)$; (2) for some $\varepsilon > 0$ and for every $t \in (t_*, t_* + \varepsilon)$ there is a point of t-level $i \le m$ in $L_{a,b}(t)$, and, for $t = t_*$, a point x of the t_*-level i is a boundary point of $L_{a,b}(t_*)$. Note that $i = m$ is impossible, because otherwise, for $t \in (t_*, t_* + \varepsilon)$, $L_{(a,b)}$ would contain 3 points of level m. So $1 \le i \le m - 1$.

Two different cases are possible.

(a) $x = t_*^{(\ell)}$ or $t_*^{(r)}$. Let, for example, $x = t_*^{(r)}$. Then $\sigma_2^i(t_*^{(r)}) = \sigma_2^m(t_*^{(r)}) = t_*$. Then $\sigma_2^{m-i}(t_*) = t_*$, i.e. t_* is an interior point of $L_{a,b}(t_*)$ of a level $< m$, hence, for **all** t close to t_* there is a point of the same t-level in $L_{a,b}$. Contradiction.

(b) $x = t_*^a$ or t_*^b. Let, for example, $x = t_*^a$ be a point of the level $i < m$. Then

$$\sigma_2^{2m-i}(t_*) = t_*.$$

Set $y = \sigma_2^m(t_*)$. Then y is a point of t_*-level $m - i$, $1 \le m - i \le m - 1$. Because of the condition (4.1), we get that $y \in L_{a,b}(t_*)$. Hence a point of the same level $m - i$ preserves in some neighborhood of t_*. Contradiction.

The proof of Theorem 4.1 proceeds as follows. Let, for some $c_1 \in \mathbb{C} \setminus M$ and slope τ_1, $t_{c_1}^{\tau_1} \in T$. Join c_1 and c_0 by a closed arc $\ell \subset \mathbb{C} \setminus M$ and find a continuous function $\tau(c) \in [\tau_1, \pi/2]$, with $\tau(c_0) = \pi/2$, $\tau(c_1) = \tau_1$, such that $t_c^{\tau(c)} \in T$ if $c \in \ell$. By Lemmas 4.1 and 4.2, for every $c \in \ell$ and corresponding $\tau = \tau(c)$, the open intervals

$$I_c^a = \{\omega = t + ih \colon h = \frac{h_c}{d}, (t_c^\tau)^{(\ell)} < t < (t_c^\tau)^a\},$$

$$I_c^b = \{\omega = t + ih \colon h = \frac{h_c}{d}, (t_c^\tau)^b < t < (t_c^\tau)^{(r)}\}$$

belong to comb-domain H_τ of f_c. Then we can apply the map $\Phi_c = (B_c^\tau)^{-1} \circ p \colon H_\tau \to (A_c)_\tau$, and obtain that the following subsets of $\Gamma(\frac{h_c}{d})$

$$\Gamma_1 = \Phi_c(I_c^a) \quad \text{and} \quad \Gamma_2 = \Phi_c(I_c^b),$$

are curves and change continuously as $c \in \ell$. For $c = c_0$, Γ_1 and Γ_2 are arcs of the boundary of the component of $\operatorname{int} K(h_{c_0}/d)$, which contains $\alpha_1(c_0)$. Therefore, for every $c \in \ell$, Γ_1 and Γ_2 are the arcs of the boundary of the component $K_1(c)$ of $\operatorname{int} K(h_c/d)$, which contains $\alpha_1(c)$.

On the other hand, for some $\nu \in (0,1)$

$$\theta^-(\nu) \le t_c^\tau \le \theta^+(\nu),$$

and $\Lambda^\nu \subset \overline{L_{a,b}(t_c^\tau)}$. It follows that the ray R_θ^τ, with $\theta = \theta^-(\nu)$, lands inside the component $K_1(c)$ together with all iterations $f^{mk}(R_\theta^\tau)$, $k = 1, 2, 3, \ldots$. Let z_0 be the landing point of R_θ^τ, and let $g_1 = f^{-m} \colon K(h_c) \to K_1(c)$ be a branch of f^{-m} such that $\alpha_1(c) = \bigcap_{k=0}^\infty g_1^k(K_1(c))$. We have proved that

$$z_0 \in g_1^k(K_1(c)),$$

for every $k = 0, 1 \ldots$. Thus, $z_0 = \alpha_1(c)$ and $\Lambda^\nu \subset \Lambda_\tau(\alpha_1(c))$. Theorem 4.1 is proved. Part (A) of Theorem 1.1 corresponds to the case $\tau = \pi/2$ in Theorem 4.1.

5. Angle of access and the basic inequality

In this section we consider an arbitrary repelling fixed point a of the polynomial $P = f^m$, where $f = f_c$ and $c \in \mathbb{C} \setminus M$. Note that the result of this section, Theorem 5.1, and its proof, hold without any changes for a repelling fixed point of any nonlinear polynomial P.

5.1. A TORUS. Consider a branch $g = P^{-1}$, such that $g(a) = a$, defined in a small disc D_ϵ around the point a. The action of g defines a torus S if, for every x, we identify the points $g^n(x)$, $n \in \mathbb{N}$.

Let us linearize the map $g: D_\epsilon \to D_\epsilon$ around its attracting point a by a univalent function F which is holomorphic in D (the **Königs** coordinate function):

$$F \circ g(z) = \frac{1}{\lambda} F(z),$$

with $\lambda = P'(a)$, $F(a)$=0, and $F'(a) = 1$. We will use definitions from [P]. Let $W = F(D)$, $\psi = F^{-1}$: $W \to D$ and $\tilde{W} = \exp^{-1}(W)$. Denote

$$\tilde{\psi}: \tilde{W} \to D, \quad \tilde{\psi}(z) = \psi \circ \exp(z).$$

Suppose L is any logarithm of the multiplier λ of P, i.e. $\exp(L) = \lambda$. Then $\tilde{\psi}$ conjugates translation by L and P. Moreover, the torus S is conformally equivalent to $\mathbb{C}/\Pi$, with $\Pi = L \circ \mathbb{Z} \times 2\pi i \circ \mathbb{Z}$. Let Γ: $\mathbb{C} \to \mathbb{C}/\Pi \simeq S$ denote the corresponding canonical covering. Let γ: $\mathbb{T} \to S$ be a non-trivial (i.e. not homotopic to a point) Jordan curve in S. Then there exist $p, q \in \mathbb{Z}$, with $(p,q) = 1$, such that any lifting $\tilde{\gamma}$ of γ by Γ satisfies

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) + q \cdot L - 2\pi i \cdot p.$$

The number $\sigma = qL - p \cdot 2\pi i$ does not depend on the choice of generator L. Changing the orientation of γ if necessary, we can suppose that $q \geq 0$. If $q = 0$, then $p = \pm 1$. Further, if $q > 0$ then changing L by $+2\pi i$ the number p is changed to $p + q$. Thus for suitable choice of L we have $p \in \{0, 1, \ldots, q-1\}$. With this normalization the number p/q is called the **combinatorical rotation number of the closed curve** $\gamma \subset S$. We denote by $\Gamma_{p,q}$ the family of the curves on S with the same combinatorical rotation number p/q, $q \geq 1$.

5.2. THE BASIC INEQUALITY. The rotation number $\nu_\tau(a)$ and the rotation set $\Lambda_\tau(a)$ of the point a are well defined for every slope $\tau \in (0, \pi)$.

Let us fix the slope τ and consider the corresponding comb Q_τ.

We shall now define the **angle of access to the point** a. Let $\theta \in \Lambda_\tau(a)$. Then there exists a τ-ray R_θ^τ that lands at the point a; if R_θ^τ contains a cut, then it is either the right R_θ^+ or the left R_θ^- limit ray. Suppose the rotation number of a is rational, that is, $\nu_\tau(a) = p/q$. Then θ is a point of a cycle $\bar{\theta}$ of σ_2. We will use notation from Lemma 2.1 (the first property of the hedgehog).

Definition 5.1: The angle of access to the cycle $\bar{\theta}$ is the value $\gamma(\bar{\theta})$, if R_θ does not contain a τ-cut; otherwise, it is either $\gamma^{(r)}(\bar{\theta})$ or $\gamma^{(\ell)}(\bar{\theta})$ depending on whether the right or the left ray lands at a.

The angle of access to a fixed point a is the sum $\phi_\tau(a)$ of the angles of access to all different cycles of σ_d, $d = 2^m$, in $\Lambda_\tau(a)$.

The main result of this section is the following

THEOREM 5.1 (see also [L1]): *Let, for some τ, $\nu_\tau(a) = \frac{p}{q}$ be rational (in reduced form). Then, for a branch of* $\log\lambda$ *of the multiplier $\lambda = P'(a)$,*

$$\log\lambda \in \left\{ z\colon \left| z - \left(2\pi i\frac{p}{q} + \frac{\pi \log d}{q\phi_\tau(a)} \right) \right| < \frac{\pi\log d}{q\phi_\tau(a)} \right\}.$$

Proof of the theorem: (cf. [Y], [L2], [P], [LS]) Let $\theta \in \Lambda_\tau(a)$ and C be the angle of access to θ with the top θ (i.e. C is $W_\tau^{(r)}(\theta)$, $W_\tau^{(\ell)}(\theta)$ or $W_\tau(\theta)$, see Lemma 2.1). The angle of access C is invariant under the map σ_d^q. Define two families of curves $\tilde{E}$ and E. We take $\tilde{E} = \{\tilde{e}_\alpha\}$ to be the family of all intervals in the angle C such that the interval $\tilde{e}_\alpha$ joins a point V, $\operatorname{Im} V = y$, with the point $\theta + (V - \theta)d^{-q}$. Here $y > 0$ is small and fixed and α is the angle between $\tilde{e}_\alpha$ and $\mathbb{R}$. The family $\tilde{E}$ projects by $\Phi\colon H_\tau \to A_\tau$ to a family of curves in a small disk centered at a, and after that to the family E of curves on the torus S. Any two curves $e_1, e_2 \in \Gamma$ are disjoint, because the level y is small and the point θ is periodic of period q. Moreover, the curves $e \in \Gamma$ in the torus are closed. The torus S is conformally equivalent to $\mathbb{C}/\Pi$, where $\Pi = \log\lambda\cdot\mathbb{Z}\times 2\pi i\cdot\mathbb{Z}$. Every e lies in $\Gamma_{p,q}$, i.e. it lifts to a curve γ in $\mathbb{C}$ which joins a point z with $z + q\log\lambda - p2\pi i$, with some choice of $\log\lambda$. This is because the set $\bar{\theta}$ is a rotation set of σ_d with rotation number p/q, i.e. exactly p curves among $\{\gamma + k\log\lambda\}_{k=0}^{q-1}$ in $\mathbb{C}/2\pi i\mathbb{Z}$ are disposed between γ and $\gamma + \log\lambda$ (including γ).

The listed geometric properties of $\tilde{E}$ and E lead to the following estimates. First, introduce the metric ρ on the torus S, which is induced by the Euclidean one using representation $S \cong \mathbb{C}/\Pi$. In its turn, the metric ρ induces a metric in a punctured neighbourhood of the point a, and then a metric $\tilde{\rho}$ in the angle C (with the help of the map Φ^{-1}).

Now let $M = AL^{-2}$, where A is the area of the set of the points $z \in e$, $e \in E$, with respect to the metric ρ, and L is the infimum of lengths of the $e \in E$, with respect to the metric ρ. The number $\tilde{M} = \tilde{A}\tilde{L}^{-2}$ is defined similarly, but for the family $\tilde{E}$ and the metric $\tilde{\rho}$. Then $M = \tilde{M}$ since the metrics ρ and $\tilde{\rho}$, and the families E and $\tilde{E}$ are obtained by a (anti-) holomorphic isomorphism, see [Ah]. Now we use standard estimates in the method of extremal length [Ah] to obtain

$$\frac{\phi(\bar{\theta})}{q\log d} \leq \tilde{M} = M \leq \frac{A}{|q\log\lambda - 2\pi ip|^2}.$$

Summing up these inequalities over all the periodic orbits $\bar{\theta}$ in $\Lambda_\tau(a)$, we come to the following inequality:

$$\frac{\phi_\tau(a)}{q\log\lambda} \leq \frac{2\pi\log|\lambda|}{|q\log\lambda - 2\pi ip|^2},$$

where the equality is attained if and only if the metric $\tilde{\rho}$ is logarithmic one, which is impossible if the Julia set is not an analytic arc.

6. Proof of Theorem 1.1 (B)

Given j and a point $t_c + ih_c \in \Omega_j$, choose a slope τ such that $t_c^\tau = \arg_\tau(t_c + ih_c) \in (\theta^-_{a,b}(p_j/q_j), \theta^+_{a,b}(p_j/q_j))$. For every $V \in \Omega_j$, $V \notin \mathbb{R}$, denote by $\varphi(V)$ the angle of vision of the interval $(\theta^-_{a,b}(p_j/q_j), \theta^+_{a,b}(p_j/q_j))$ from the point V.

According to Theorem 5.1 (with $d = 2^m$), Theorem 4.1, and Remark 2.2, it is enough to show the following:

$$\inf_{V\in\Omega_j} \varphi(V) \geq \phi(mq_j, m(q_{j+1} - q_j)), \tag{6.1}$$

where the function $\phi(\alpha,\beta)$ is bounded from below by an absolute positive constant, and $\phi(\alpha,\beta)$ tends to $\pi/4$ as α and β tend to infinity. If we have that, then we let

$$\aleph = \frac{\pi\log 2}{\phi}$$

and obtain part (B) of Theorem 1.1.

Let V_1 and V_2 be two vertices of Ω_j with the imaginary part $h^{(j)}$. Then, for $V \in \Omega_j$,

$$\varphi(V) \geq \min_{1\leq i\leq 2} \varphi(V_i).$$

Now the angles $\varphi(V_i)$ are calculated explicitly using Proposition 3.1, part I(ii). Details are left to the reader.

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